Cusp forms & Hecke operators for congruence subgroups

Let $q \in \mathbb{Z}_{>0}$ and let

$$\Gamma(q) := \{ \gamma \in SL_2(\mathbb{Z}) = \Gamma : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod q \}$$

which is called the principal congruence subgroup of level q

$\Gamma(q) \trianglelefteq SL_2(\mathbb{Z})$ since we have the exact sequence

$$1 \longrightarrow \Gamma(q) \longrightarrow SL_2(\mathbb{Z}) \xrightarrow{\bmod q} SL_2(\mathbb{Z}/q\mathbb{Z}) \longrightarrow 1$$

Explain exact sequence

where the only non trivial part is surjectivity (Exercise)

Then $[SL_2(\mathbb{Z}) : \Gamma(q)] = \# SL_2(\mathbb{Z}/q\mathbb{Z}) = q^3 \prod_{p \mid q} (1 - \frac{1}{p^2})$

proof of $=$ If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}/q\mathbb{Z})$,

then $ad - bc + hq = 1 \Rightarrow (c, d, q) = 1$

and for each row $(c \;\; d)$, $\exists\, q$ solutions to $ad - bc \equiv 1 (q)$,

since $\boxed{\text{wlog}}$ $(d, q) = 1 \Rightarrow a \equiv (1 + bc) d^{-1} \bmod q$

all values of b ie q ones

Hence $|SL_2(\mathbb{Z}/q\mathbb{Z})|$

$$= q \, |\{(c,d) \bmod q \text{ st } (c,d,q) = 1\}|$$

eliminate (c,d) st $r \mid q$ and $r \mid c$ and d
Inclusion–Exclusion

$$= q \left(q^2 - \sum_{p \mid q} \left(\frac{q}{p}\right)^2 + \sum_{p_1 p_2 \mid q} \left(\frac{q}{p_1 p_2}\right) - \sum_{p_1 p_2 p_3 \mid q} \left(\frac{q}{p_1 p_2 p_3}\right) + \cdots \right)$$

$$= q \sum_{r \mid q} \mu(r) \left(\frac{q}{r}\right)^2$$

Moebius

r runs over SF divisors with a sign

recall $\sum_{r \mid n} f(r)$ is a mult. fct when f is since for $(n_1, n_2) = 1$, there is a bijection between $d \mid n_1 n_2$ and pairs (d_1, d_2) st $d_1 \mid n_1$ & $d_2 \mid n_2$

Then $q \sum_{r \mid q} \mu(r) \left(\frac{q}{r}\right)^2 = q \prod_{p \mid q} f(1) + f(p) + f(p^2) + \cdots + f(p^{v_p(q)})$

with $f(r) = \mu(r) \left(\frac{q}{r}\right)^2$

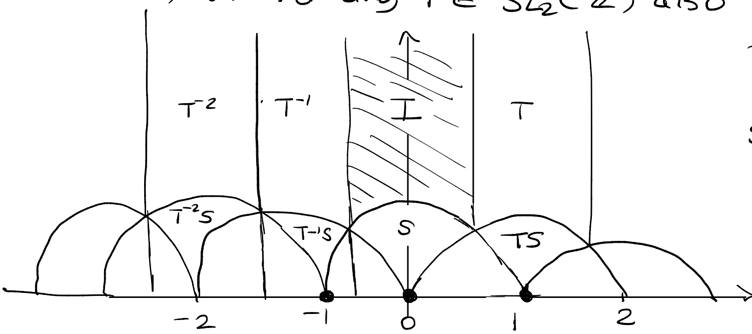
$$= q \prod_{p \mid q} \underbrace{q^2}_{f(1)} - \underbrace{\left(\frac{q}{p}\right)^2}_{f(p)} = q^3 \prod_{p \mid q} \left(1 - \frac{1}{p^2}\right) \qquad \blacksquare$$

Cusp of $\Gamma(q)$

Back to $\Gamma = SL_2(\mathbb{Z})$.

$\mathcal{F} = \{ z : |z| > 1 \;\&\; -\frac{1}{2} < x < \frac{1}{2} \}$ is a fundamental domain

Of course, $\gamma \mathcal{F}$ for any $\Gamma \in SL_2(\mathbb{Z})$ also



$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

roywilliams.github.io/play/js/sl2z

Image of ∞ in each fundamental domain $\quad \gamma\infty = \dfrac{a\infty + b}{c\infty + d} = \begin{cases} \dfrac{a}{c} & c \neq 0 \\ \infty & c = 0 \end{cases}$

$S(\infty) = 0$, $TS(\infty) = 1$

$T^{-1}S(\infty) = -1$ etc

We then define $\quad \text{cusp} = \{\infty\} \cup \{\frac{c}{d}\} = \mathbb{P}^1(\mathbb{Q})$

and $\gamma \in SL_2(\mathbb{Z})$ acts on $\mathbb{P}^1(\mathbb{Q})$ by $\gamma t = \dfrac{at+b}{ct+d}$

For any $\Gamma \leq SL_2(\mathbb{Z})$,

the set of inequivalent cusps in $\Gamma \backslash \mathbb{P}^1(\mathbb{Q})$

Then, there is only one equivalent cusp for $SL_2(\mathbb{Z})$

which is ∞ is we take $\mathcal{F}$,

or 0 if we take $S\mathcal{F}$ etc

lemma Let $(a,b) = 1$, $(c,d) = 1$, $\begin{pmatrix} a \\ b \end{pmatrix} \equiv \begin{pmatrix} c \\ d \end{pmatrix} \bmod q$. Then $\exists \gamma \in \Gamma(q)$ st $\begin{pmatrix} a \\ b \end{pmatrix} = \gamma \begin{pmatrix} c \\ d \end{pmatrix}$

proof $\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ cusp at ∞. Then $a \equiv 1 \,(q) \Rightarrow \exists u, v$ st $au - bv = \dfrac{(1-a)}{q}$ (since $(a,b) = 1$)

and $\gamma = \begin{pmatrix} a & qv \\ b & 1+qu \end{pmatrix} \in \Gamma(q)$ is the matrix st $\gamma \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

when $\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

In general, take integers r, s st $cr + ds = 1$ and $\tau = \begin{pmatrix} c & -s \\ d & r \end{pmatrix}$. Then $\tau\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} \equiv \begin{pmatrix} a \\ b \end{pmatrix} \bmod q$

or $\tau^{-1}\begin{pmatrix} a \\ b \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bmod q \Rightarrow \exists\, \gamma$ st $\gamma\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \tau^{-1}\begin{pmatrix} a \\ b \end{pmatrix}$ by above

$$\Rightarrow \tau\gamma\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \tau\gamma\underbrace{\tau^{-1}\begin{pmatrix} c \\ d \end{pmatrix}}_{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad \blacksquare$$

In fact, we can prove that if $\frac{a}{b}$ and $\frac{c}{d}$ equivalent under $\Gamma(q)$

ie $\exists\, \gamma \in \Gamma(q)$ st $\gamma\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\Leftrightarrow \pm\begin{pmatrix} a \\ b \end{pmatrix} \equiv \begin{pmatrix} c \\ d \end{pmatrix}$ (q)

If q is odd, then $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \notin \Gamma_0(q)$, and then it suffice to count

$$\#\{(c,d) \bmod q : (c,d,q) = 1\} = \#\text{ of inequivalent cusps of } \Gamma(q)$$

$$= q^2 \prod_{p \mid q} (1 - p^{-2}) \text{ by the above.}$$

Def Any subgroup of $SL_2(\mathbb{Z})$ containing $\Gamma(q)$ is called a <u>congruence subgroup of level q</u>. For example

$$\Gamma_0(q) = \{ \gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod q \} \quad \text{Hecke congruence subgroup}$$

$$\Gamma_1(q) = \{ \gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod q \}$$

Prop A set of representatives for $\Gamma_0(q) \backslash SL_2(\mathbb{Z}) = \dot{\bigcup}_{r \in R \subseteq SL_2(\mathbb{Z})} \Gamma_0(q) r$ is given by

$$\begin{pmatrix} * & * \\ u & v \end{pmatrix} \in SL_2(\mathbb{Z})$$

with $v \mid q$ and $0 < u \leq \frac{q}{v}$

Hence the index is

$$V_q = [SL_2(\mathbb{Z}) : \Gamma_0(q)] = q \prod_{p \mid q} (1 + \tfrac{1}{p})$$

Remark The notation $*\ *$ means once you have u and v as described (which are coprime), choose any m, n st $mu + nv = 1$ and take $\begin{pmatrix} m & n \\ u & v \end{pmatrix}$

Proof

$$\underbrace{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}_{\Gamma_0(q)} \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{SL_2(\mathbb{Z})} = \begin{pmatrix} * & * \\ \gamma a + \delta c & \gamma b + \delta d \end{pmatrix} =: \begin{pmatrix} * & * \\ u & v \end{pmatrix}$$

$$\boxed{\gamma \equiv 0\,(q)}$$

also coprime to q since $\Gamma_0(q) \subseteq SL_2(\mathbb{Z})$

and $\gcd(\gamma b + \delta d, q) = \gcd(\delta d, q) = \gcd(d, q)$
$= \gcd(v, q)$

and by taking a suitable $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, we can have $\boxed{v = \gamma b + \delta b = (d, q)}$ ie $v \mid q$

All solutions of $\gamma' b + \delta' d = v$ are given by $\gamma' = \gamma + dt$, $\delta' = \delta - bt$ (one parameter family)

for t being any value $t \equiv 0(\frac{q}{v})$, to insure that $\gamma' \equiv 0(q)$.

This sends u to $u' = u + t$, so we can change u mod $\frac{q}{v}$.

For the index

$$[SL_2(\mathbb{Z}) : \Gamma_0(q)] = \sum_{v \mid q} \underbrace{\left| \{ u \ (\bmod \tfrac{q}{v}) : (u, v) = 1 \} \right|}$$

$$= \prod_{p \mid q} \sum_{\substack{v \mid p^\alpha \\ \alpha = v_p(q)}} \underbrace{p^\alpha}_{\substack{v=1 \\ u \bmod p^\alpha \\ \text{st } (u,1)=1}} + \underbrace{(p^{\alpha-1} - p^{\alpha-2})}_{\substack{v=p \\ u \bmod p^{\alpha-1} \\ \text{st } (u,p)=1}} + (p^{\alpha-2} - p^{\alpha-3}) + \dots + \underbrace{(p-1)}_{v=p^{\alpha-1}} + \underbrace{1}_{\substack{v=p^\alpha \\ u \bmod 1 \\ \text{st } (u,p^\alpha)=1}}$$

$$= \prod_{p \mid q} p^{v_p(q)} (1 + \tfrac{1}{p}) = \boxed{q \prod_p (1 + \tfrac{1}{p})}$$

Fundamental Domain

Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$, and R a set of coset representatives for $\Gamma \backslash SL_2(\mathbb{Z})$. Then the set

$$D_\Gamma = \sum_{\gamma \in R} \gamma(D), \text{ where } D \text{ is a fundamental domain for } SL_2(\mathbb{Z})$$

is a f D for Γ, ie $\forall z \in \mathcal{H}$, $\exists \gamma \in \Gamma$ st $\gamma(z) \in D_\Gamma \cup \partial D_\Gamma$

Moreover, γ is unique up to multiplication of an element of $\Gamma \cap \{\pm 1\}$, except possibly if $\gamma z \in \partial D_\Gamma$

Proof $z \in \mathcal{H}$ and $z_0 \in D$, $\gamma_0 \in SL_2(\mathbb{Z})$ st $z = \gamma_0 z_0$

where $\gamma_0 = \gamma^{-1} r$, $\gamma \in \Gamma$, $r \in R$

$$\Rightarrow \gamma z = \gamma \gamma_0 z_0 = \gamma \gamma^{-1} r z_0 = r z_0 \in D_\Gamma.$$

Suppose that $\gamma_1, \gamma_2 \in \Gamma$ and $\gamma_1 z$, $\gamma_2 z \in D_\Gamma$. To show $\gamma_1 = u \gamma_2$, $u \in \{\pm 1\} \cap \Gamma_0(q)$

Then $\exists\, r_1, r_2 \in R$ st $\gamma_i z \in r_i D \Leftrightarrow (r_i)^{-1} \gamma_i z \in D$

If $\gamma_i z \notin \partial D_\Gamma$, then $(r_i)^{-1} \gamma_i z \notin \partial D$, and since Stab$(z)$ in D is $\left\{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$

while Stab$(i) = \langle S \rangle$
Stab(ω), Stab$(\omega+1)$
are $\langle ST \rangle$, $\langle TS \rangle$
order 6

$$\Rightarrow (r_2)^{-1} \gamma_2 \gamma_1^{-1} r_1 \in \{\pm I\}$$

$$\Leftrightarrow \gamma_2 \gamma_1^{-1} = \pm r_2 r_1^{-1} \in \langle R \rangle \cap \Gamma$$
$$= \{\pm 1\} \cap \Gamma$$

Fundamental Domain

Then, a fundamental domain for $\mathcal{H}/\Gamma_0(q)$ is $\bigcup_{\gamma \in R} \gamma(F)$, for any set of repn for $\Gamma \backslash SL_2(\mathbb{Z})$.

and $|R| = V_q = q \prod_{p \mid q} (1 + \frac{1}{p})$

Ex $\Gamma_0(2)$ $V_2 = 2\,(1 + \frac{1}{2}) = 3$ (u, v) st $v \mid q$ and $u \bmod \frac{q}{v}$ & coprime

representatives $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \right\}$

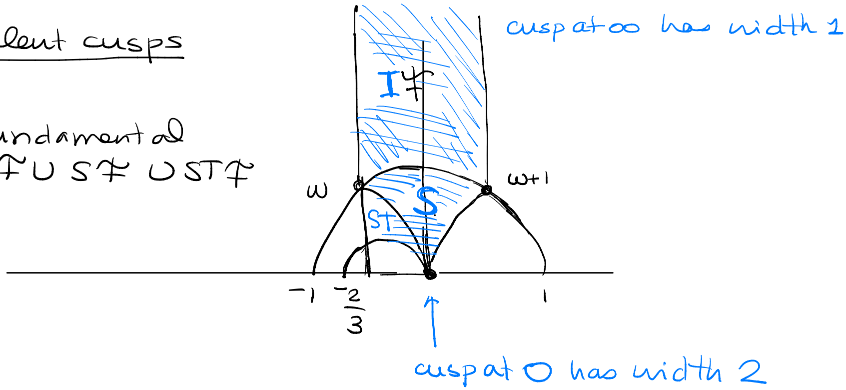
$\infty \mapsto \infty$ $\quad \infty \mapsto 1$ $\quad \infty \mapsto 0$

other representaties $\left\{ \underset{I}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \underset{S}{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}, \underset{ST}{\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}} \right\}$

$\infty \mapsto \infty$ $\quad \infty \to 0$ $\quad \infty \to 0$

2 equivalent cusps

Another fundamental region: $\mathcal{F} \cup S\mathcal{F} \cup ST\mathcal{F}$



Prop 2.6 A set of inequivalent cusps for $\Gamma_0(q)$ is given by

$$\frac{u}{v} \quad \text{with } v \mid q,\ (u,v)=1,\ \underbrace{u \bmod (v, \tfrac{q}{v})}_{\text{some collapsing.}}$$

and the number of inequivalent cusps is

$$h = \sum_{vw=q} \varphi((v,w)) \qquad \text{where } \varphi \text{ is the Euler } \varphi\text{-fct}$$

$vw = q \to v \mid q$ and $\frac{q}{v} = w$

proof Write the representatives $\tau = \begin{pmatrix} * & * \\ u & v \end{pmatrix}$

as $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\tau = \begin{pmatrix} u & * \\ v & * \end{pmatrix}$. Then $\begin{pmatrix} u & * \\ v & * \end{pmatrix}(\infty) = \frac{u}{v}$ are cusps

Then the rational points $\frac{u}{v}$ with $v \mid q$, $(u,v)=1$, $u \bmod \frac{q}{v}$ are cusps.

Suppose $\frac{u'}{v'} \sim \frac{u}{v}$ ie $\exists \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(q)$

st $\begin{pmatrix} u' \\ v' \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}_{\in \Gamma_0(q)} \begin{pmatrix} u \\ v \end{pmatrix}$ $\qquad \gamma \equiv 0\,(q)$

Then $v' = \gamma u + \delta v \implies v' \mid v$ and $v \mid v' \implies v = v'$

Also, $\delta \equiv 1 \bmod q/v$.

Now $u' = \alpha u + \beta v \equiv \alpha u \bmod v \overset{\alpha\delta \equiv 1(q)}{\equiv} \delta^{-1} u \bmod q \equiv u \bmod \frac{q}{v}$
which proves the result.

Exercise The # of such (u,v) is h.

Ex $\Gamma_0(2)$ $\quad h = \sum_{vw=2} \varphi((v,w)) = \varphi((2,1)) + \varphi((1,2)) = 2$

The spaces $M_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(q), \chi)$

① f is holomorphic on $\mathcal{H}$

② $f|_\gamma(z) := (cz+d)^{-k} f(\gamma z)$. Then $(f|_\gamma)(z) = \chi(\gamma) f$

ie $f(\gamma z) = \chi(\gamma)(cz+d)^k f(z)$ $\boxed{\text{for } \gamma \in \Gamma_0(q)}$

③ f is holomorphic at the cusps, which are the orbits of $\Gamma_0(q)$ acting on $\mathbb{P}^1(\mathbb{Q})$ we need to explain what it means.

Now, χ is a Dirichlet character mod q ie

ie $\chi : (\mathbb{Z}/q\mathbb{Z})^* \longrightarrow \mathbb{C}^*$ extended to $\mathbb{Z}$

by $\chi(a) = \chi(a \bmod q)$ $\quad (a,q)=1$

$\chi(a) = 0$ $\quad (a,q) \neq 1$

Also, $\chi(\gamma) = \overline{\chi}(a)$ where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Since $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma_0(q)$, we have $f(z) = \overline{\chi(-1)}\,(-1)^k f(z)$,

so we also take $\boxed{\chi(-1) = (-1)^k}$ if not the spaces are empty

Wenow explain ③

Let $\mathfrak{a}$ be a cusp and $t \in \mathbb{P}^1(\mathbb{Q})$ a representative.

$\Gamma_0(q)_t = \{ \gamma \in \Gamma_0(q) : \gamma t = t \}$

Take $\boxed{\gamma_t \in SL_2(\mathbb{Z}) \text{ st } \gamma_t \infty = t}$. Then $\gamma \in \Gamma_0(q)_t \Leftrightarrow \gamma_t^{-1} \gamma \gamma_t \in SL_2(\mathbb{Z})_\infty$

and $\gamma_t^{-1} \Gamma_0(q)_t \gamma_t \leqslant SL_2(\mathbb{Z})_\infty = \left\langle \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$

$$\Rightarrow \gamma_t^{-1} \Gamma_0(q)_t \gamma_t = \left\{ \pm \begin{pmatrix} 1 & b m_t \\ & 1 \end{pmatrix} : b \in \mathbb{Z} \right\} := H_{\mathfrak{a}} \quad ⊛$$

All this does not depend on representative, and we could write $\Gamma_{\mathfrak{a}}$ etc. The group ⊛ is called $H_{\mathfrak{a}} \subseteq SL_2(\mathbb{Z})_\infty$, and $m_{\mathfrak{a}}$ = width of the cusp.

Interpretation $\quad \mathcal{F}_{\Gamma_0(q)} = \displaystyle\mathring{\bigcup_{r \in R_q}} r(\mathcal{F})$

The number of copies $r(\mathcal{F})$ which touch each cusp $\mathfrak{a}$ is the same for equivalent cusps and it is $m_{\mathfrak{a}}$

Let $\mathfrak{a}$ be a cusp of width $m_{\mathfrak{a}}$. Choose $t \in \mathfrak{a}$ and $\gamma_t \in SL_2(\mathbb{Z})$, $\gamma_t \infty = t$

Claim $(f|_{\gamma_t})$ is invariant under that action of $H_{\mathfrak{a}}$.

Let $\gamma_{\mathfrak{a}} = \begin{pmatrix} 1 & m_{\mathfrak{a}} \\ 0 & 1 \end{pmatrix}$. generator of $H_{\mathfrak{a}}$ Then $\gamma_t \gamma_{\mathfrak{a}} \gamma_t^{-1} = \tilde{\gamma}_{\mathfrak{a}} \in \Gamma_0(q)_t$ by (*)

$$(f|_{\gamma_t})|_{\gamma} = f|_{\gamma_t \gamma} = f|_{\tilde{\gamma}\gamma_t} = (f|_{\tilde{\gamma}})|_{\gamma_t} = \chi(\tilde{\gamma}) f|_{\gamma_t}$$

and $\chi^{-1}(\tilde{\gamma}_{\mathfrak{a}}) f|_{\gamma_{\mathfrak{a}}}$ is periodic with period $m_{\mathfrak{a}}$.

Remark $\chi^{-1}(\tilde{\gamma}_{\mathfrak{a}}) \neq 0$ as $\tilde{\gamma} \in \Gamma_0(q) \Rightarrow \tilde{\gamma} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $c \equiv 0\ (q)$ $\Rightarrow (a, q) = 1$

Write $\chi^{-1}(\tilde{\gamma}_{q}) = e(K_{\mathfrak{a}})$ $\quad 0 \leq K_{\mathfrak{a}} < 1$

Then $e(-K_{\mathfrak{a}} z) f|_{\gamma_{\mathfrak{a}}}(z)$ is periodic of period $h_{\mathfrak{a}}$

$e(-K_{\mathfrak{a}} z) f|_{\gamma_{\mathfrak{a}}}$ can be expressed as a Laurent series in $q_{\mathfrak{a}} = \exp\left(\frac{2\pi i z}{h_{\mathfrak{a}}}\right)$, as we did before for the cusp ∞

$$(f|_{\gamma_{\mathfrak{a}}})(z) = e(K_{\mathfrak{a}} z) \sum_{n \geq -N}^{\infty} a_f(n)\, q_{\mathfrak{a}}^n$$

$f \in M_k(\Gamma_0(q), \chi)$ $\quad (f|_{\gamma_{\mathfrak{a}}})(z) = e(K_{\mathfrak{a}} z) \sum_{n=0}^{\infty} a_f(n)\, q_{\mathfrak{a}}^n$

$f \in S_k(\Gamma_0(q), \chi)$ $\quad (f|_{\gamma_{\mathfrak{a}}})(z) = \sum_{n=1}^{\infty} a_{\mathfrak{a}}(n)\, q_{\mathfrak{a}}^n$

In fact, the condition for cusp forms is needed only for $K_{\mathfrak{a}} \neq 0$ since we already have exp decay when $K_{\mathfrak{a}} \neq 0$

<u>Remark</u> By using the <u>scaling matrix</u> $\sigma_{\mathfrak{a}}$ of the cusp $\in SL_2(\mathbb{R})$, related to $\begin{pmatrix} \sqrt{m_{\mathfrak{a}}} & 0 \\ 0 & 1/\sqrt{m_{\mathfrak{a}}} \end{pmatrix}$, we can get that the generating series at each cusp has period 1 <u>ie</u> can be written as $\sum_{n=1}^{\infty} a(n)\, e(nz)$

<u>Hecke Operators for $M_k(\Gamma_0(q),\chi)$ & $S_k(\Gamma_0(q),\chi)$</u>

For $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\chi(\rho) := \overline{\chi(a)}$. Then, $\chi(\rho) = 0$ if $(a,q) \neq 1$, and

$$T_n(f) = n^{\frac{k}{2}-1} \sum_{\rho \in \Delta_n} \bar{\chi}(\rho)\, f|_\rho \qquad \Delta_n = \text{representatives for } G_n/\Gamma = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad = n,\ 0 \leq b < d \right\}$$

$$= n^{\frac{k}{2}-1} \sum_{\rho \in \Delta_n^q} \bar{\chi}(\rho)\, f|_\rho \qquad \text{where } \Delta_n^q = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Delta_n : (a,q) = 1,\ ad = n,\ 0 \leq b < d \right\}$$

<u>Recall</u> $\Delta_n \times \Gamma \longleftrightarrow \Gamma \times \Delta_n$

$\forall \rho \in \Delta_n, \tau \in \Gamma\ \exists \rho', \tau'$ <u>st</u> $\rho\tau = \tau'\rho'$

gives $\Delta_n^q \times \Gamma_0(q) \longleftrightarrow \Gamma_0(q) \times \Delta_n^q$

(can be shown by looking at the formulas for explicit intertwining in Iwaniec)

<u>Thm 6.16</u> The Hecke operator $T_n = T_n^{\chi, k}$ is <u>st</u>:

$$T_n : M_k(\Gamma_0(q),\chi) \longrightarrow M_k(\Gamma_0(q),\chi)$$
$$S_k(\Gamma_0(q),\chi) \longrightarrow S_k(\Gamma_0(q),\chi)$$

<u>proof</u> Same as before, using the intertwining above, and exp decay is preserved by slash operator. ∎

$$\boxed{T_n T_m = \sum_{d \mid (m,n)} \chi(d)\, d^{k-1}\, T_{\frac{mn}{d^2}}}$$

The multiplicativity does not depend on where the operators act, and still true. Hence, all Hecke operators are generated by

$$T_p(f) = p^{\frac{k}{2}-1} \sum_{ad=p} \sum_{0 \le b < d} \chi(a)\, f\left(\frac{az+b}{d}\right) d^{-k}\, p^{k/2}$$

$$= \frac{1}{p} \sum_{ad=p} \chi(a)\, a^k \sum_{0 \le b < d} f\left(\frac{az+b}{d}\right)$$

$$= \frac{1}{p} \sum_{0 \le b < p} f\left(\frac{z+b}{p}\right) + \chi(p)\, p^{k-1} f(pz)$$

or $T_p = \frac{1}{p} \sum_{0 \le b < p} \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} + \chi(p)\, p^{k-1} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$

$\boxed{\text{If } p \mid q}$ then $\chi(p) = 0$ and $T_p = \frac{1}{p} \sum_{b \bmod p} \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}$

which is why we need the character

and using $T_m T_n = \sum_{d \mid (m,n)} \chi(d)\, d^{k-1}\, T_{\frac{mn}{d^2}}$

we get $T_p T_p = T_{p^2}$ <u>etc</u>, <u>ie</u> $(T_{p^\nu}) = (T_p)^\nu$

<u>Thm</u> Let $k > 2$, $\chi(-1) = (-1)^k$. For all $m \ge 0$ and $n \ge 1$, we have

$$\boxed{T_n P_m = \sum_{d \mid (m,n)} \chi\left(\frac{n}{d}\right) \left(\frac{n}{d}\right)^{k-1} P_{\frac{mn}{d^2}}}$$

<u>Q</u> What do we mean by $P_m = P_m^{\chi,k}$ here?

Poincaré series for $\Gamma_0(q)$ with χ

$$P_m(z) = \sum_{\Gamma_\infty \backslash \Gamma_0(q)} \overline{\chi}(\tau)\, j_\tau(z)^{-k}\, e(m\tau z)$$

$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right\}$

proof As before with obvious modifications. Exercise.

$m=0$ $\quad E_k(z) = \sum_{\Gamma_\infty \backslash \Gamma_0(q)} \overline{\chi}(\tau)\, j_\tau(z)^{-k}$ is an Eigenfct

& all T_n with eigenvalue $\sigma_{k-1}(n,\chi) = \sum_{d|n} \chi(d)\, d^{k-1}$

$m \geq 1$ and $n | q^\infty$ (same prime factors),

then all divisors of (m,n) are not coprime with q, except the divisor 1 so for $\chi(\frac{n}{d})$ in $\boxed{\text{RHS}}$ to be non zero, we $\frac{n}{d} = 1$ ie $d | (m,n)$ contains the value n ie $\boxed{n | m}$.

Then if $m \geq 1$ and $n | q^\infty$ and $n | m$

$$T_n P_m = P_{m/n}$$

If $(n,q)=1$, then $\chi(\frac{n}{d}) = \chi(n)\chi(d)^{-1}$ since $(d,q)=1$, and

$$T_n P_m = \chi(n)\, n^{k-1} \sum_{d | (m,n)} \overline{\chi}(d)\, d^{1-k}\, P_{\frac{mn}{d^2}}$$

As before since the formula is almost symmetric in m, n.

Cor If $m,n \geq 1$ and $\boxed{(mn,q)=1}$, then

$$\chi(m)\, m^{k-1}\, T_n P_m = \chi(n)\, n^{k-1}\, T_m P_n$$

We now need the adjoint operator such that

$$\langle T_n f, g \rangle = \langle f, T_n g \rangle$$

We could redo the proofs that we did for $SL_2(\mathbb{Z})$ and get

$$\langle T_n f, g \rangle = \chi(n) \langle f, T_n g \rangle \quad \text{if } (n,q)=1$$

But then, we will get a result only for the cusp forms in $S_k(\Gamma_0(q), \chi)$ which are generated by the Poincaré series $P_m(z)$ with $\boxed{(m,q)=1}$, which are not all the cusp forms.

But this can be fixed (see Iwaniec), and we get

Thm If $(n,q)=1$ and $\boxed{f, g \in S_k(\Gamma_0(q), \chi)}$ all of them

then $\langle T_n f, g \rangle = \chi(n) \langle f, T_n g \rangle = \langle f, \overline{\chi(n)}\, T_n g \rangle$

ie the adjoint of T_n is $T_n^* = \overline{\chi}(n) T_n$

Then T_n is normal (commutes with its adjoint), and $\overline{\chi}(n)^{1/2} T_n$ is self adjoint.

Remark Applying Thm with an eigenfunction $f = g \neq 0$, we get

$\langle T_n f, f \rangle = \lambda_n \langle f, f \rangle$ and $\langle f, \overline{\chi}(n) \lambda(n) f \rangle = \chi(n) \overline{\lambda(n)} \langle f, f \rangle$

$\Rightarrow \lambda(n) = \chi(n) \overline{\lambda(n)}$ $(n,q)=1$

Thm In the space of cusp forms $S_k(\Gamma_0(q), \chi)$, there exists an orthonormal basis $\mathcal{F}$ which consists of eigenfunctions for all the Hecke operators T_n with $(n,q)=1$ (as before linear algebra)

New forms and old forms

$f \in S_k(\Gamma_0(q), \chi)$ Hecke Eigenform

ie $T_n f = \lambda(n) f$ $(n,q)=1$.

Then if $f(z) = \sum_{m=1}^{\infty} a(m) e(mz)$, then

$$\lambda(n) a(m) = \sum_{d \mid (m,n)} \chi(d) d^{k-1} a\left(\frac{mn}{d^2}\right) \quad \text{for } (n,q)=1$$

For $m=1$ $\lambda(n) a(1) = a(n)$, $(n,q)=1$. If $a(1) \neq 0$, then we can normalise st $a(1) = 1$ and $\lambda(n) = a(n)$ for all $(n,q)=1$ so the $a(n)$ have a lot of arithmetic from the T_n. (multiplicative!)

$\boxed{\text{BUT}}$ $a(1) = 0 \not\Rightarrow f = 0$ ie it is possible that the Fourier expansion of f is only supported on $a(n)$ with $(n,q)>1$

Those are the old forms associated with non primitive (induced from a lower level $q' \mid q$) characters χ.

Take χ be a character for modulus q which is not primitive
ie χ is periodic mod q' for $q' \mid q$

Ex $\chi_p = \left(\frac{\cdot}{p}\right)$ Legendre symbol is primitive mod p.

It is also a non primitive character modulo p^2 or p^k (exactly the same character) or mod pr for any $r \in \mathbb{Z}_{\geq 1}$ (differs at $(n,r)\neq 1$).

Def The conductor of (χ modulo q) is the smallest integer $d \geq 1$ st χ is periodic mod d ie $\chi(m) = \chi(n)$ for $m \equiv n \ (d)$ and $(n,q) = (m,q) = 1$

Let χ be a character mod q which is primitive mod $q^* \mid q$, and $q', d \in \mathbb{Z}_{\geq 1}$ st $q^* \mid q'$ and $q'd \mid q$. $\boxed{\text{Think } q' = q^*}$

Let χ' mod q' be the character mod q' induced by χ, ie $\chi'(\alpha) = \chi(\alpha)$ $(\alpha, q') = 1$

Then $f(z) \in S_k(\Gamma_0(q'), \chi') \Longrightarrow f(dz) \in S_k(\Gamma_0(q), \chi)$

proof

$$f(dz)\Big|_{\gamma} = f\Big|_{\left(\begin{smallmatrix} d & 0 \\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)} = f\Big|_{\left(\begin{smallmatrix} \alpha & \beta d \\ \gamma/d & \delta\end{smallmatrix}\right)\left(\begin{smallmatrix} d & 0 \\ 0 & 1\end{smallmatrix}\right)}$$

$\gamma \in \Gamma_0(q)$, $\in \Gamma_0(q)$, $\in \Gamma_0(q')$ since $\frac{\gamma}{d} \equiv 0 \left(\frac{q}{d}\right)$ $\Longrightarrow \left(\alpha, \frac{q}{d}\right) = 1$

$= \chi(\alpha)$ since $(\alpha, q') = 1$

$$= \chi'(\alpha) f\Big|_{\left(\begin{smallmatrix} d & 0 \\ 0 & 1\end{smallmatrix}\right)} \Rightarrow f(dz) \in S_k(\Gamma_0(q), \chi)$$

Now if $f(z) = \sum_{m=1}^{\infty} a(m)\, e(mz)$ (at some cusp),

then $f(dz) = \sum_{m=1}^{\infty} a(m)\, e(mdz) = \underbrace{\sum_{m \equiv 0 (d)}}_{\text{ie } (m,q) > 0 \text{ always}} a\left(\frac{m}{d}\right) e(mz)$

Such f are called old forms.

If $S_k^b(\Gamma_0(q), \chi)$ is the linear space spanned by old forms, then let $S_k^{\#}(\Gamma_0(q), \chi)$ be the orthogonal space (w/r to Peterson's inner product).

Thm $S_k(\Gamma_0(q),\chi) = S_k^b(\Gamma_0(q),\chi) \oplus S_k^{\#}(\Gamma_0(q),\chi)$
and both spaces are stable under the Hecke operators
T_n with $(n,q)=1$

proof One can check directly that $T_n : S_k^b \longrightarrow S_k^b$,
and since $T_n^* = \overline{\chi}(n) T_n$ (the adjoint),
if $f \in S^b$ and $g \in S^{\#}$

$$0 = \langle T_n f, g\rangle = \langle f, T_n^* g\rangle \Rightarrow T_n^* g \in (S_k^b)^{\perp} = S_k^{\#}$$

$= 0$ since $T_n f$
$\in S^b$

Hecke Eigen cusp forms for primitive χ

Then $S_k^{\#}(\Gamma_0(q),\chi) = S_k(\Gamma_0(q),\chi)$

Thm If $f \in S_k(\Gamma_0(q),\chi)$ is such that $a(n)=0$
for $(n,q)=1$, then $f=0$!

proof

Write $f|_\tau = \chi(\tau) f$ for $\tau = \begin{pmatrix} a & * \\ c & d\end{pmatrix} \in \Gamma_0(q)$

$$f\left(\frac{az+b}{cz+d}\right)(cz+d)^{-k} = \chi(d) f(z)$$

and replace z by $\frac{z-d}{c}$ to get

$$z^{-k} f\left(\frac{a}{c} - \frac{1}{cz}\right) = \chi(d) f\left(\frac{z-d}{c}\right) = \chi(d) f\left(\frac{z}{c} - \frac{d}{c}\right)$$

and writing the Fourier expansions at both sides give

$$z^{-k}\sum_{m=1}^{\infty} a(m)\, e\left(\frac{am}{c} - \frac{m}{cz}\right) = \chi(d)\sum_{n=1}^{\infty} a(n)\, e\left(\frac{nz}{c} - \frac{dn}{c}\right)$$

with $0 < c \equiv 0\,(q)$ (to avoid dividing by 0...)
and $ad \equiv 1\,(c)$ (determinant 1)

15.

Using $c=q$ and summing over $a \bmod q$, $(a,q)=1$

where $ad \equiv 1\,(q) \Rightarrow \bar{a} \equiv d\,(q)$ $\rightarrow$ inverse mod q

RHS
$$\sum_{n=1}^{\infty} a(n) \underbrace{\sum_{\substack{(a,q)=1\\ a \bmod q}} \overline{\chi}(a)\, e\left(\frac{-\bar{a}n}{q}\right)}\, e\left(\frac{nz}{q}\right)$$

$$\sum_{\substack{(a,q)=1\\ a \bmod q}} \chi(a)\, e\left(\frac{-an}{q}\right)$$

If $(n,q)=1$ $\quad b=-an$ gives

$$\sum_{\substack{(b,q)=1\\ b \bmod q}} \chi\left(\frac{b}{-n}\right) e\left(\frac{b}{q}\right) = \overline{\chi}(-n) \underbrace{\sum_{b \bmod q} \chi(b)\, e\left(\frac{q}{b}\right)}_{\text{Gauss sum } \tau(\chi)}$$

means $\chi(b)\overline{\chi}(-n)$

If χ is primitive, this is also true for $(n,q)>1$ ie for all n. Exercise

$$\Rightarrow \sum_{n=1}^{\infty} a(n)\, \overline{\chi}(-n)\, \tau(\chi)\, e\left(\frac{nz}{q}\right) = 0$$

since $\quad a(n)=0 \quad (n,q)=1$ hypothesis

$\qquad \chi(-n)=0 \quad (n,q)>1$ char of modulus q

and RHS $=0$. By unicity of Fourier expansion, this implies that LHS vanishes, where

$$\text{LHS} = z^{-k} \sum_{m=1}^{\infty} a(m)\, e\left(\frac{-m}{qz}\right) \sum_{\substack{(a,q)=1\\ a \bmod q}} e\left(\frac{am}{q}\right)$$

where $\boxed{c_q(m) = \sum_{\substack{(a,q)=1 \\ a \bmod q}} e\left(\frac{am}{q}\right)}$ Ramanujan sum

Exercise If $\mu^2(q) = 1$, then

$$c_q(m) = \mu\left(\frac{q}{(m,q)}\right) \varphi((m,q))$$

and in particular $c_q(m) \neq 0 \ \forall m$

$\Longrightarrow a(m) = 0$ for all m !!!

If q is not SF, more involved argument (Iwaniec)

Remark If χ is primitive of order ℓ, then cond(χ) is SF away from ℓ.

Ex Primitive char of order 2 mod p^k, $k \geq 1$, $p \neq 2$

$\chi = \left(\frac{\cdot}{p}\right)$ Legendre symbol of conductor p

Primitive characters of order 2 mod 2^k, $k \geq 1$

$$\chi_4(n) = \begin{cases} 1 & n \equiv 1 \ (4) \\ -1 & n \equiv 3 \ (4) \end{cases}$$

Cor if $f \neq 0$ an Eigenfunction of all Hecke operators st $(n,q) = 1$, then $a(1) \neq 0$, since $a(1) = 0$

and $a(n) = a(1)\lambda(n) \quad \forall (n,q) = 1$

implies $a(n) = 0 \ \forall (n,q)=1 \implies f = 0$ by Thm.

Also from Thm, f is determined uniquely by the Eigenvalues of T_n for $(n,q)=1$ (Multiplicity one principle) using $f-g$ for f in Thm ∎

Then, $T_n(T_m(f)) = T_n(T_m f) = \lambda(n) T_m(f)$

for $(n,q)=1$

$\implies T_m(f)$ has the same EV as f for $(n,q)=1$

$\implies T_m(f) = \lambda(m) f$!!

This leads to

Thm If $f \in S_k(\Gamma_0(q), \chi)$ is an eigenfct for all T_n st $(n,q)=1$, then f is an eigfct for all T_m, ie

$$T_m(f) = \lambda(m) f$$

Besides the Hecke operators, we need the Fricke (or Atkin-Lehner) involution:

Def $W f = f|_\omega$ where $\omega = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}$ $\notin \Gamma_0(q)$ since det $= q$

$$= \det(\omega)^{k/2} (qz)^{-k} f(-1/q)$$

$$= q^{k/2} q^{-k} z^{-k} f(-1/q)$$

$$= q^{-k/2} z^{-k} f(-1/q)$$

Since $\omega^2 = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix} = \begin{pmatrix} -q & 0 \\ 0 & -q \end{pmatrix}$

$$\Rightarrow W^2(f) := f\big|_{\omega^2} = q^k (-q)^{-k} f(z)$$
$$= (-1)^k f(z)$$

Moreover $\omega \Gamma_0(q) = \Gamma_0(q)\omega$

ie $\omega\tau = \tau'\omega$ where $\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\tau' = \begin{pmatrix} d & -c/q \\ -bq & a \end{pmatrix}$

Then for $f \in S_k(\Gamma_0(q), \chi)$ and $\tau \in \Gamma_0(q)$,

$$(f|_\omega)|_\tau = f\big|_{\omega\tau} = f\big|_{\tau'\omega} = \chi(\tau') f\big|_\omega$$
$$= \overline{\chi}(\tau) f|_\omega$$

and $W : S_k(\Gamma_0(q), \chi) \longrightarrow S_k(\Gamma_0(q), \overline{\chi})$

Furthermore $\underbrace{\langle Wf, Wg\rangle}_{\text{in } S_k(\Gamma_0(q),\overline{\chi})} = \underbrace{\langle f, g\rangle}_{\text{in } S_k(\Gamma_0(q),\chi)}$

Thm If $(n,q)=1$, then

$$W T_n^{\chi} = \chi(n)\, T_n^{\overline{\chi}}\, W$$

Proof $$W T_n^{\chi}(f) = n^{\frac{k}{2}-1} \sum_{\rho \in \Delta_n} \overline{\chi}(\rho)\, f\big|_{\rho\omega}$$

$$T_n^{\overline{\chi}} W(f) = n^{\frac{k}{2}-1} \sum_{\rho \in \Delta_n} \chi(\rho)\, f\big|_{\omega\rho}$$

and more intertwining.
(see Iwaniec) $\blacksquare$

To deal with χ & $\overline{\chi}$, define the complex conjugation operator K

$$(Kf)(z) := \overline{f}(-\overline{z})$$

If $f(z) = \sum a(n)\, e(nz)$,

then $(Kf)(z) = \sum \overline{a(n)}\, \overline{e(-n\overline{z})} = \sum \overline{a(n)}\, e(nz)$

since $\overline{e^{-2\pi i n(x-iy)}} = \overline{e^{-2\pi i n x - 2\pi n y}} = e^{-2\pi n y} e^{2\pi i n x}$
$= e^{2\pi i (nx + iny)} = e(nz)$

Properties ① $K\lambda f = \overline{\lambda} Kf \qquad \lambda \in \mathbb{C}$

② $K^2 = 1, \quad WK = (-1)^k KW$

③ $K : S_k(\Gamma_0(q), \chi) \longrightarrow S_k(\Gamma_0(q), \overline{\chi})$

④ $\langle Kf, Kg \rangle = \langle f, g \rangle$

⑤ $K T_n^{\chi} = T_n^{\overline{\chi}} W$ for all n

$\boxed{\text{Set } \overline{W} = KW}$. Then

① $\overline{W}^2 = 1, \quad \overline{W} \eta f = \overline{\eta}\, \overline{W} f \quad$ for $\eta \in \mathbb{C}$

② $\overline{W} : S_k(\Gamma_0(q), \chi) \longrightarrow S_k(\Gamma_0(q), \chi)$ $\boxed{\text{now good}}$

③ $T_n \overline{W} = \chi(n)\, \overline{W}\, T_n \quad$ if $(n, q) = 1$

Now suppose that $f \in S_k(\Gamma_0(q), \chi)$, $f \neq 0$, is an Eigen function for all T_n with eigenvalues $\lambda(n)$ for $(n,q)=1$.

Then
$$T_n \overline{W} f = \chi(n) \overline{W} T_n(f)$$
$$= \chi(n) \overline{W} \lambda(n) f$$
$$= \chi(n) \overline{\lambda(n)} \overline{W} f$$
$$\overset{(6.57)}{=} \lambda(n) \overline{W} f \qquad (n,q)=1$$

ie $\overline{W} f$ is an eigenfunction of T_n with eigenvalues same as f for $(n,q)=1$

$\Rightarrow \overline{W} f = \eta f$ by multiplicity one.

Thm If f is a Hecke cusp form in $S_k(\Gamma_0(q), \chi)$, then f is also an eigenfunction of $\overline{W}$

ie $\overline{W} f = \eta f$.

Furthermore, $|\eta| = 1$.

proof We just proved the statement except the furthermore:

By the above properties
$$f = \overline{W}^2 f = \overline{W} \eta f = \bar{\eta} \overline{W} f = \bar{\eta} \eta f$$
$\Rightarrow |\eta \bar{\eta}| = 1$.

Thm The eigenvalue of the involution $\overline{W}$ for $f \in S_k(\Gamma_0(q), \chi)$ a normalized Hecke eigen cusp form is given by
$$\eta = \tau(\bar{\chi}) \lambda(q) q^{-k/2}$$

proof Again some matrix decomposition
$$\begin{pmatrix} 1 & u/q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1/q \\ q & 0 \end{pmatrix} \begin{pmatrix} 1 & v/q \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} u & (uv-1)q \\ q & v \end{pmatrix}}_{\in \Gamma_0(q)}$$

where we choose $uv \equiv 1 \ (q)$

st the matrix M on RHS $\in \Gamma_0(q)$.

Then
$$f\Big|\begin{pmatrix}1 & u/q\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1/q\\ q & 0\end{pmatrix} = f\Big|_{M\begin{pmatrix}1 & -v/q\\ 0 & 1\end{pmatrix}}$$
$$= \chi(v)\, f\Big|\begin{pmatrix}1 & -v/q\\ 0 & 1\end{pmatrix}$$
since $f \in S_k(\Gamma_0(q), \chi)$.

Summing over $u \bmod q$, $(u,q)=1$, we get

$$g = \sum_{\substack{u \bmod q\\ (u,q)=1}} f\Big|\begin{pmatrix}1 & u/q\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1/q\\ q & 0\end{pmatrix} \qquad uv \equiv 1\ (q)$$

$$= \sum_{\substack{v \bmod q\\ (v,q)=1}} \chi(v)\, f\Big|\begin{pmatrix}1 & -v/q\\ 0 & 1\end{pmatrix}$$

not needed because of χ

since f is normalized, $\lambda(n) = a(n)$

and using $f(z) = \sum \lambda(n)\, e(nz)$
the Fourier expansion

RHS

$$g(z) = \sum_{v \bmod q} \chi(v)\, f\left(z - \frac{v}{q}\right)$$

$$= \sum_{n=1}^{\infty} \lambda(n)\left[\sum_{v \bmod q} \chi(v)\, e\left(\frac{-nv}{q}\right)\right] e(nz)$$

$$= \sum_{n=1}^{\infty} \lambda(n)\, \tau(n,\chi)\, \chi(-1)\, e(nz)$$

$$= \chi(-1)\,\tau(\chi)\sum_{n=1}^{\infty}\lambda(n)\overline{\chi}(n)\,e(nz)$$ since χ is primitive (Ass 3)

$$\Rightarrow K g(z) = \chi(-1)\overline{\tau(\chi)}\sum_{n=1}^{\infty}\underbrace{\overline{\lambda}(n)\chi(n)}_{\begin{cases}0 & (n,q)>1\\ \lambda(n) & (n,q)=1\end{cases}} e(nz)$$

$$= \chi(-1)\overline{\tau(\chi)}\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\lambda(n)e(nz) \quad := S_q(f)$$

$$= \tau(\overline{\chi})\,S_q(f)$$

LHS $\displaystyle\sum_{u \bmod q}^{*} f\Big|\begin{pmatrix}1 & u/q\\ 0 & 1\end{pmatrix}$ use $\displaystyle\sum_{d\mid(u,q)}\mu(d) = \begin{cases}1 & (u,q)=1\\ 0 & \text{otherwise}\end{cases}$

and some matrix computations which make $W = f\Big|\begin{pmatrix}0 & -1\\ q & 0\end{pmatrix}$ appear, and then λ

we get another expression for g, and then Kg which is:

$$\boxed{\begin{aligned} K g(z) &= \eta\sum_{ad=q}\mu(a)\overline{\lambda}(d)\left(a/d\right)^{k/2} d\, f(az).\\ &= \tau(\overline{\chi})(S_q f)(z)\end{aligned}}$$

Comparing the first FC : $\tau(\overline{\chi})\lambda(1) = \eta\,\overline{\lambda(q)}\,q^{1-k/2}\lambda(1)$

$\boxed{\begin{aligned}a&=1\\ d&=q\end{aligned}}$

and the result follows. ∎